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The Ambrosetti–Prodi problem for gradient elliptic systems with critical homogeneous nonlinearity

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ABSTRACT

In this work we study the system

$$\begin{cases} -\Delta u = au + bv + F_u(u_+, v_+) + f_1(x) & \text{in } \Omega, \\ -\Delta v = bu + cv + F_v(u_+, v_+) + f_2(x) & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is bounded with smooth boundary, $N \geqslant 3$, $F = H + G$, where H is a $2^* \equiv 2N/(N-2)$ positively homogeneous function, G is a lower order perturbation, $w_+ = \max\{w, 0\}$ and $f_1, f_2 \in L^r(\Omega)$, $r > N$. Using the Mountain Pass Theorem we prove existence of two solutions. If $N = 3, 4$ and 5, an additional hypothesis over the subcritical term is needed.

1. Introduction

In this work we study the system

$$\begin{cases} -\Delta u = au + bv + H_u(u_+, v_+) + G_u(u_+, v_+) + f_1(x) & \text{in } \Omega, \\ -\Delta v = bu + cv + H_v(u_+, v_+) + G_v(u_+, v_+) + f_2(x) & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^N$, $N \geqslant 3$. Being 2^* the critical Sobolev constant given by $2N/(N-2)$, we suppose that H is a 2^* positively homogeneous C^1 function and G a subcritical perturbation which will play an important role when treating lower dimensions. We are also denoting $w_+ = \max\{w, 0\}$ and assuming $f_1, f_2 \in L^r(\Omega)$, $r > N$. The matrix A given by the constants a, b and c has eigenvalues smaller than the first eigenvalue of $-\Delta$.

This system is motivated by scalar problems that were first studied by Ambrosetti and Prodi in [1]. This pioneer work established existence, multiplicity and non-existence results for the problem $-\Delta u = g(u) + f(x)$ in Ω, $u = 0$ on $\partial\Omega$ depending heavily upon the non-homogeneous term f and on the interaction of g with the spectrum of $-\Delta$; denoting $g_- = \lim_{s\to-\infty} g(s)/s$ and $g_+ = \lim_{s\to+\infty} g(s)/s$, they supposed $0 < g_- < \lambda_1 < g_+ < \lambda_2$, where λ_k stands for the mentioned spectrum. After that, several authors have extended their results in many different ways. We refer to [7] and references therein for a better background on this problem, which is well known as of Ambrosetti–Prodi type. Since the literature and the variety of conditions that followed are very extensive, we will try to approach some of them to our case. We emphasize the case $g_- \in (0, \lambda_1)$ and $g_+ = \infty$. This situation can be treated variationally by the Mountain Pass Theorem and was studied in [6,15] for the scalar case, where subcritical g was considered. Following these papers, critical cases were investigated.

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In de Figueiredo and Jianfu [7] (see also [11] for related results) a one-sided critical growth was first considered. The authors studied $g(u) = \lambda u + u_+^{2^*-1}$, and were able to prove existence of two solutions only when $N \geqslant 7$. The problem consisted in proving that the mini–max level of the functional associated avoided the noncompactness levels, and this could be done only with restriction to the dimension. This restriction is somehow natural when considering critical growth since it is known that low dimensions may have different behaviors (see Brezis and Nirenberg [3]). More recently, using a technique introduced in [13], Calanchi and Ruf [4] improved the results of [7] to $N \geqslant 6$ and, by adding a positive subcritical term to the equation, they could discuss the cases $N = 3, 4$ and 5.

The purpose of this work is to obtain, for a gradient system of elliptic equations, some of the results achieved in [4] for the scalar case. Based also on [7] and de Morais Filho and Souto [10], we prove the existence of two solutions, one of them negative, depending on the non-homogeneous terms f_1 and f_2. The methods used here follow the ideas given in [4] but we had to pass through several technical difficulties that appeared, for example, when treating a more general critical term like H. Our primary intention is to treat this system with a critical term such as

$$H(u,v) = a_1 u^{2^*} + a_{k+1} v^{2^*} + \sum_{i=2}^{k} a_i u^{\alpha_i} v^{\beta_i}, \tag{1.2}$$

where $a_i \geqslant 0$, $k \geqslant 0$ and $\alpha_i, \beta_i > 1$ satisfy $\alpha_i + \beta_i = 2^*$. This kind of critical term seems to be first considered in [10] already for the p-Laplacian and gives an unified approach to systems with either an uncoupled critical term (as in [14]) or a coupled one (as in [9]). The result obtained here also improves those in [9] with respect to the dimension N, since there, the authors worked on an extension of [7]. We must also refer to more previous works on the Ambrosetti–Prodi problem to elliptic systems regarding other types of hypothesis, such as [5,8,12].

2. Hypothesis and main theorem

We can rewrite problem (1.1) in its vectorial and parameterized form as follows:

$$\begin{cases} -\Delta U = AU + \nabla\big(H(U_+) + G(U_+)\big) + P(x) + Te_1(x) & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $U = \binom{u}{v}$, $U_+ = \binom{u_+}{v_+}$, $A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in M_{2\times 2}(\mathbb{R})$,

$$P(x) = \begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix} \in L^r(\Omega) \times L^r(\Omega), \tag{2.2}$$

$T = (t,s)^T$ and e_1 stands for the first positive eigenfunction of $-\Delta$ with Dirichlet boundary condition, normalized in L^2 with λ_1 the corresponding eigenvalue.

For the sake of better exposition, let us denote

$$F_T(x) = P(x) + Te_1(x). \tag{2.3}$$

This parameterized non-homogeneous term plays a crucial role on existence theorems for problems like (2.1) because we discuss existence of solutions depending on the parameter $T \in \mathbb{R}^2$.

First of all, we need to establish necessary conditions over the matrix A. We suppose:

(A_1) $\det(\lambda_1 I - A) > 0$;
(A_2) $b, \lambda_1 - a, \lambda_1 - c > 0$ and $\mu_1 > 0$;

where we are denoting $\mu_1 \leqslant \mu_2$ its eigenvalues. Note that these conditions above imply $0 < \mu_1 \leqslant \mu_2 < \lambda_1$, which indicates us a Mountain Pass Theorem approach to the problem. The following inequalities are quite useful and will be used throughout the whole work:

$$\mu_1 |U|^2 \leqslant (AU, U)_{\mathbb{R}^2} \leqslant \mu_2 |U|^2, \quad \forall U \in \mathbb{R}^2, \tag{2.4}$$

where $(\,,)_{\mathbb{R}^2}$ denotes the usual inner product in $\mathbb{R}^2$.

Now we focus on the nonlinearities. We have the following assumptions for the critical part.

(H_1) $H \in C^1(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$ and $H, H_u, H_v \geqslant 0$, $H \not\equiv 0$;
(H_2) $H(\lambda u, \lambda v) = \lambda^{2^*} H(u,v)$, $\forall \lambda > 0$ (that means, H is a 2^*-positively homogeneous function);
(H_3) $H_u(0,1) = H_v(1,0) = 0$;
(H_4) $(s,t) \mapsto H(s^{1/2^*}, t^{1/2^*})$ is concave.

Remark 1. Regarding condition (H_4), it is important to say that it comes from the necessity of having a Hölder type inequality given by

$$\int_\Omega H(u,v) \leqslant H\big(\|u\|_{2^*}, \|v\|_{2^*}\big), \quad \text{for all } u, v \in L^{2^*}(\Omega),\ u, v \geqslant 0, \tag{2.5}$$

and (H_2) and (H_4) guarantee (2.5) (see [10, Proposition 4]). This is the only place where we use this condition, but it is somehow a big restriction on H: Although there are other examples of functions satisfying (H_1)–(H_3) besides polynomials like (1.2), they were the only examples we could find in order to fulfill all the conditions we needed to have. Nevertheless they are as generic as possible if we are trying to apply the techniques used in this work.

Concerning the lower order term, we also ask for homogeneity, but of subcritical degree. We will not require a condition like (H_4) for G.

(G_1) $G \in C^1(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+)$ and $G, G_u, G_v \geqslant 0$;
(G_2) There exists $2 < p < 2^*$ such that $G(\lambda u, \lambda v) = \lambda^p G(u,v)$, $\forall \lambda > 0$ (that means, G is a p-positively homogeneous function);
(G_3) $G_u(0,1) = G_v(1,0) = 0$.

Remark 2. Let us observe here that (H_3), (G_3) allow us to redefine H and G in the whole plane, letting $H(u,v) = H(u_+, v_+)$, $G(u,v) = G(u_+, v_+)$ and we will still have $H, G \in C^1(\mathbb{R}^2)$. Therefore, we are always considering H and G as these extensions.

We will look for solutions in $E = H_0^1(\Omega) \times H_0^1(\Omega)$ equipped with its usual norm

$$\big\|(u,v)\big\|^2 = \|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2.$$

That is, $U = (u,v) \in E$ is a (weak) solution to problem (2.1) if

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega \nabla v \nabla \psi - \int_\Omega \big(A(u,v), (\varphi,\psi)\big)_{\mathbb{R}^2} - \int_\Omega \big(\nabla H(U_+), (\varphi,\psi)\big)_{\mathbb{R}^2}$$
$$- \int_\Omega \big(\nabla G(U_+), (\varphi,\psi)\big)_{\mathbb{R}^2} - \int_\Omega \big(F_T(x), (\varphi,\psi)\big)_{\mathbb{R}^2} = 0 \quad \text{for all } (\varphi,\psi) \in E.$$

Let us define a partial order in $\mathbb{R}^2$: $(t_1, s_1) < (t_2, s_2)$ if $t_1 < t_2$ and $s_1 < s_2$. Now we are able to state our main result, which will be proved in the last section.

Theorem 1. *Suppose (A_1)–(A_2), (H_1)–(H_4) and (G_1)–(G_3). If $N \geqslant 6$ there exist two lines $\alpha_1(t)$ and $\alpha_2(t)$ with negative slopes such that if*

$$(t,s) \in \big\{(\tau,\theta) \in \mathbb{R}^2;\ \theta < \alpha_1(\tau),\ \theta < \alpha_2(\tau)\big\},$$

then problem (2.1) has at least two solutions, one of which is negative. Moreover, if $N = 3, 4, 5$ and we also suppose

(G_4) $G(U) > 0$ if $|U_+| > 0$ and $p/2^* > 2/3(1 + 1/N)$,

then the same result holds.

Remark 3. Note that we are not discarding $G = 0$ for $N \geqslant 6$ but we have to do so for lower dimensions, which are considered admitting also (G_4).

Remark 4. The hypothesis (A_1)–(A_2), (H_1)–(H_4) and (G_1)–(G_3) allow us to apply the results obtained in [8], which guarantee the existence of a Lipchitzian curve Γ in $\mathbb{R}^2$ splitting the plane into two disjoint unbounded regions $\mathbb{R}^2 = R_1 \cup \Gamma \cup R_2$ such that problem (2.1) has

1. no solution if $T \in R_1$,
2. at least one solution if $T \in R_2$.

In our case, we do not know what happens in Γ since a priori estimates seem to be needed in order to prove existence of solutions and critical growth imposes several difficulties when trying to get these estimates. The solution obtained in R_1 is guaranteed by a sub-supersolution method. We refer to [8] for the details.

Denoting $S=\{(\tau,\theta)\in\mathbb{R}^2;\ \theta<\alpha_1(\tau),\ \theta<\alpha_2(\tau)\}$, it is obvious that the Lipchitzian curve ∂S lies under Γ and $S\subset R_2$.

3. The negative solution and restatement of the problem

The lines $\alpha_1(t)$, $\alpha_2(t)$ are determined by the boundary of the region where we can find a negative solution. Once we have a nonpositive solution in hand, we can obtain a second one by modifying the original problem around it. We define them in the following way:

Let $\Phi_0=(\phi_0,\psi_0)$ be the solution of the linear system

$$\begin{cases}-\Delta U=AU+P(x) & \text{in } \Omega,\\ U=0 & \text{on } \partial\Omega\end{cases}$$

and consider a 2×1 matrix $\mu(T)$ such that $\mu(T)e_1$ solves

$$\begin{cases}-\Delta U=AU+Te_1(x) & \text{in } \Omega,\\ U=0 & \text{on } \partial\Omega.\end{cases}$$

A straightforward calculation shows that

$$\mu(T)=\frac{1}{\det(\lambda_1 I-A)}\begin{pmatrix}(\lambda_1-c)t+bs\\ bt+(\lambda_1-a)s\end{pmatrix}.$$

Defining

$$\phi_T=\frac{(\lambda_1-c)t+bs}{\det(\lambda_1 I-A)}e_1+\phi_0$$

and

$$\psi_T=\frac{bt+(\lambda_1-a)s}{\det(\lambda_1 I-A)}e_1+\psi_0,$$

it is obviously seen that $\Phi_T=(\phi_T,\psi_T)$ satisfies

$$\begin{cases}-\Delta \Phi_T=A\Phi_T+F_T(x) & \text{in } \Omega,\\ U=0 & \text{on } \partial\Omega,\end{cases} \tag{3.1}$$

where F_T is defined in (2.3).

Now we must look for parameters T such that Φ_T is nonpositive, so it will also be a solution to (2.1).

Recall that $P\in L^r(\Omega)\times L^r(\Omega)$ with $r>N$. Thus $\Phi_0\in C^{1,\nu}\times C^{1,\nu}$ for some $0<\nu<1$ by regularity arguments. Then

$$\left\|\frac{\det(\lambda_1 I-A)}{(\lambda_1-c)t+bs}\phi_T-e_1\right\|_{C^1}=\left\|\frac{\det(\lambda_1 I-A)}{(\lambda_1-c)t+bs}\phi_0\right\|_{C^1}$$

and

$$\left\|\frac{\det(\lambda_1 I-A)}{bt+(\lambda_1-a)s}\psi_T-e_1\right\|_{C^1}=\left\|\frac{\det(\lambda_1 I-A)}{bt+(\lambda_1-a)s}\psi_0\right\|_{C^1}.$$

Let $\varepsilon>0$ be such that if $\|\phi-e_1\|_{C^1}<\varepsilon$ then $\phi>0$; in order to have $\phi_T,\psi_T<0$ we need the following inequalities:

$$(\lambda_1-c)t+bs<0;$$
$$bt+(\lambda_1-a)s<0;$$
$$\left|(\lambda_1-c)t+bs\right|>(\varepsilon b)^{-1}\det(\lambda_1 I-A)\|\phi_0\|_{C^1};$$
$$\left|bt+(\lambda_1-a)s\right|>\left[\varepsilon(\lambda_1-a)\right]^{-1}\det(\lambda_1 I-A)\|\psi_0\|_{C^1},$$

which are satisfied if

$$s<\left(\frac{c-\lambda_1}{b}\right)t-(\varepsilon b)^{-1}\det(\lambda_1 I-A)\|\phi_0\|_{C^1} \tag{3.2}$$

and

$$s<\left(\frac{b}{a-\lambda_1}\right)t-\left[\varepsilon(\lambda_1-a)\right]^{-1}\det(\lambda_1 I-A)\|\psi_0\|_{C^1}. \tag{3.3}$$

These two lines in the right side of (3.2) and (3.3) are the desired $\alpha_1(t)$ and $\alpha_2(t)$.

From now on, let $T = (t, s) \in \{(\tau, \theta) \in \mathbb{R}^2;\ \theta < \alpha_1(\tau),\ \theta < \alpha_2(\tau)\}$.

Consider the problem

$$\begin{cases} -\Delta V = AV + \nabla H\big((V + \Phi_T)_+\big) + \nabla G\big((V + \Phi_T)_+\big) & \text{in } \Omega, \\ V = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Since Φ_T is a negative solution of (2.1), it is easily seen that if V is a nontrivial solution to problem (3.4) then $U = V + \Phi_T$ will solve problem (2.1) and we will be done.

Thus, we must look for nontrivial solution to problem (3.4). In [7] and afterwards in [9,14], critical point theorems were directly used. In these works, the authors reached a technical difficult when trying to prove that the *PS*-sequence obtained in fact converged to a nontrivial solution and could overcome it by considering only $N \geqslant 7$. We propose here the idea and technique used in [4]. We shall separate the supports of the negative solution and the so-called Talenti function (that is usually used when we study elliptic problems by critical growth) in order to make some estimates easier to handle. This is done by cutting a small hole into the function Φ_T near $\partial\Omega$ and concentrating the support of the Talenti function inside this hole. This approximation creates some errors but they are easily estimated.

We begin by taking $m \in \mathbb{N}$ sufficiently large so that we can find $x_m \in \Omega$ such that $B_{4/m}(x_m) \subset \Omega$ and $|\Phi_T(x)| \leqslant C/m$ for all $x \in B_{4/m}(x_m)$ and for some $C > 0$. Note that if m_0 is such a choice, then any $m > m_0$ can also be chosen. Consider then $\eta_m \in C^\infty(\mathbb{R}^N)$ such that: $0 \leqslant \eta_m \leqslant 1$, $|\nabla \eta_m(x)| \leqslant 2m$ and

$$\eta_m(x) = \begin{cases} 0 & \text{if } x \in B_{1/m}(x_m), \\ 1 & \text{if } x \in \Omega \backslash B_{2/m}(x_m). \end{cases}$$

Now, take $\binom{\varphi_T^m}{\psi_T^m} := \Phi_T^m := \eta_m \Phi_T$ and consider

$$F_T^m = \begin{pmatrix} f_1^m \\ f_2^m \end{pmatrix}$$

satisfying

$$-\Delta \Phi_T^m = A\Phi_T^m + F_T^m \quad \text{in } \Omega. \tag{3.5}$$

Finally, consider the problem

$$\begin{cases} -\Delta U = AU + \nabla H\big((U + \Phi_T^m)_+\big) + \nabla G\big((U + \Phi_T^m)_+\big) + F_T - F_T^m & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Note that $\Phi_T - \Phi_T^m$ is trivially a solution for this problem and if $U \neq \Phi_T - \Phi_T^m$ is another solution, then it is straightforward to show that $V = U - \Phi_T + \Phi_T^m$ solves (3.4). Our goal then is to prove that there exists such U.

4. Preliminary results

We must estimate the errors created in the approximation problem (3.6), namely $\Phi_T - \Phi_T^m$ and $F_T - F_T^m$. The following lemma has the results needed to continue.

Lemma 1. *For all m sufficiently large, there exist $c_1, c_2 > 0$ such that*

$$\big\|\Phi_T - \Phi_T^m\big\| \leqslant c_1 m^{-N/2}; \tag{4.1}$$

and for all $\Psi = (\varphi, \psi) \in E$,

$$\left|\int_\Omega (f_1 - f_1^m)\varphi\right| + \left|\int_\Omega (f_2 - f_2^m)\psi\right| \leqslant c_2 \|\Psi\| m^{-N/2}. \tag{4.2}$$

Proof. For (4.1), we have to prove that $\|\varphi_T - \varphi_T^m\|_{H_0^1} \leqslant cm^{-N/2}$ and $\|\psi_T - \psi_T^m\|_{H_0^1} \leqslant cm^{-N/2}$. But this is exactly the scalar case that was proved in [4] and we refer to there for the details.

For (4.2), one only need to see that since $\Phi_T - \Phi_T^m$ solves problem (3.6), then for each $\varphi \in H_0^1(\Omega)$ we have

$$\left|\int_\Omega (f_1 - f_1^m)\varphi\right| = \left|\int_\Omega \nabla\big(\varphi_T - \varphi_T^m\big)\nabla\varphi - a\int_\Omega \big(\varphi_T - \varphi_T^m\big)\varphi - b\int_\Omega \big(\psi_T - \psi_T^m\big)\varphi\right|$$

and therefore, Hölder inequality, embedding theorems and (4.1) give us

$$\left|\int_\Omega (f_1 - f_1^m)\varphi\right| \leqslant c\|\varphi\|_{H_0^1} m^{-N/2}.$$

The same can be done also for $|\int_\Omega (f_2 - f_2^m)\psi|$ and we are done. $\square$

Let us point out some important properties of homogeneous functions. Let $\alpha \geqslant 1$ and L be a differentiable α-positively homogeneous function defined in $\mathbb{R}^2$, i.e., $L(\lambda s, \lambda t) = \lambda^\alpha L(s,t)$ for all $\lambda \geqslant 0$. Then it satisfies

- (Euler's Lemma) For all s and $t \in \mathbb{R}$,

$$sL_s(s,t) + tL_t(s,t) = \alpha L(s,t). \tag{4.3}$$

- Let

$$M_L = \sup\{|L(s,t)|:\ |s|^\alpha + |t|^\alpha = 1\}. \tag{4.4}$$

  Then

$$|L(s,t)| \leqslant M_L(|s|^\alpha + |t|^\alpha) \tag{4.5}$$

  and there exists $(s_0, t_0) \in \mathbb{R}^2$ such that

$$\begin{aligned} &|s_0|^\alpha + |t_0|^\alpha = 1, \\ &|L(s_0, t_0)| = M_L. \end{aligned} \tag{4.6}$$

- L_u and L_v are $\alpha - 1$ homogeneous.

Now, given $\epsilon > 0$ consider the function

$$u_\epsilon(x) = \left[\frac{\sqrt{N(N-2)}\epsilon}{\epsilon^2 + |x|^2}\right]^{(N-2)/2}.$$

It is known that it realizes the best Sobolev embedding constant $H^1(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$ given by

$$S = \inf_{u \neq 0} \frac{\|u\|^2_{H_0^1}}{\|u\|^2_{2^*}}. \tag{4.7}$$

Let us define also the following constant

$$S_H = \inf_{(u,v)\in E\setminus\{0\}} \frac{\|(u,v)\|^2}{(\int_\Omega H(u,v))^{2/2^*}}. \tag{4.8}$$

This is possible because of (4.5) applied to H.

Consider the 2-homogeneous function given by

$$\overline{H}(u,v) = H(u,v)^{2/2^*}. \tag{4.9}$$

The following result gives a relation between S and S_H which will be used in the proof to the main theorem.

Lemma 2. *(See de Morais Filho and Souto [10].) Let $\overline{H}$ be given in* (4.9) *and $M_{\overline{H}}$ as in* (4.4)*. If H satisfies* (2.5) *then*

$$S_H = \frac{1}{M_{\overline{H}}} S.$$

To conclude this section, take $\zeta_m \in C_0^\infty(B_{1/m}(x_m), [0,1])$ a cut-off function such that $\zeta_m = 1$ in $B_{1/2m}(x_m)$ and $\|\zeta_m\|_\infty \leqslant 4m$ and make

$$u_\epsilon^m(x) = \zeta_m(x) u_\epsilon(x - x_m).$$

We will need some estimates due to Brezis–Nirenberg:

Lemma 3. *(See Brezis and Nirenberg [3].) Fix $m \in \mathbb{N}$. Then,*

1. $\|u_\epsilon^m\|^2_{H_0^1} = S^{N/2} + O(\epsilon^{N-2})$;

2. $\|u_\epsilon^m\|_{2^*}^{2^*} = S^{N/2} + O(\epsilon^N)$;
3. $\|u_\epsilon^m\|_2^2 = K\epsilon^2 + O(\epsilon^{N-2})$ *if* $N \geqslant 5$;
4. $\|u_\epsilon^m\|_k^k \geqslant K\epsilon^{N-(N-2)k/2}$ *for all* $\epsilon \leqslant 1/2m$ *and* $k \geqslant 1$.

For $m \to \infty$ *and* $\epsilon = o(1/m)$, *we also have* (*see* [13])

5. $\|u_\epsilon^m\|_{H_0^1}^2 = S^{N/2} + O((\epsilon m)^{N-2})$;
6. $\|u_\epsilon^m\|_{2^*}^{2^*} = S^{N/2} + O((\epsilon m)^N)$.

We refer to [4] for a proof of item (4).

5. Proof of Theorem 1

First of all, we need to set the variational structure of problem (3.6). Let

$$J(U) = \frac{1}{2}\|U\|^2 - \frac{1}{2}\int_\Omega (AU, U)_{\mathbb{R}^2} - \int_\Omega H\big((U + \Phi_T^m)_+\big) - \int_\Omega G\big((U + \Phi_T^m)_+\big) - \int_\Omega \big(F_T - F_T^m, U\big)_{\mathbb{R}^2}.$$

We must look for critical points of the Functional J. And this will be done by proving some geometric properties of this functional which will satisfy the geometric hypothesis of the Mountain Pass Theorem. Since we have critical growth, J does not satisfy the *PS* condition and we have to prove that the mini–max level avoids the noncompactness levels.

Lemma 4. *There exist* $r, \delta > 0$ *such that*

$$J(V) \geqslant \delta \quad \text{for all } V \in \partial B_r(0).$$

Proof. Letting $V = (v, w) \in E$ and recalling inequalities (2.4), (4.5) for H and G and (4.2), we have

$$\begin{aligned} J(V) &\geqslant \frac{1}{2}\left(1 - \frac{\mu_2}{\lambda_1}\right)\|V\|^2 - \int_\Omega H(v_+, w_+) - \int_\Omega G(v_+, w_+) - Cm^{-N/2}\|V\| \\ &\geqslant C\|V\|^2 - M_H\left(\int_\Omega v_+^{2^*} + \int_\Omega w_+^{2^*}\right) - Cm^{-N/2}\|V\| - M_G\left(\int_\Omega v_+^p + \int_\Omega w_+^p\right) \\ &\geqslant C\|V\|^2 - C\|V\|^{2^*} - C\|V\|^p - Cm^{-N/2}\|V\|. \end{aligned}$$

Since we can control the lower order term by taking large m, the lemma is proved. Moreover, it is important to observe that these $r, \delta > 0$ are independent of large m. $\square$

Now we have to look for $W \in E$ and $R > 0$ such that $\|RW\| > r$ and $J(RW) \leqslant 0$. At this point, it is standard in the scalar problems to choose u_ϵ as such function, since it will be in its direction that we will be able to bring down the level of the functional. In our case we are working with a generic critical term, and the relation between S and S_H given in Lemma 2 suggests us to take $W = (\gamma u_\epsilon^m, \kappa u_\epsilon^m)$ where $\gamma, \kappa \geqslant 0$ are any constants such that

$$\begin{aligned} &\gamma^2 + \kappa^2 = 1, \\ &\overline{H}(\gamma, \kappa) = M_{\overline{H}}, \end{aligned} \tag{5.1}$$

where $\overline{H}$ and $M_{\overline{H}}$ are given in (4.9) and (4.4) applied to $\overline{H}$.

Lemma 5. *Let* r *be given by Lemma* 4. *There exists* $R > 0$ *such that* $\|R(\gamma u_\epsilon^m, \kappa u_\epsilon^m)\| > r$ *and*

$$J\big(R\big(\gamma u_\epsilon^m, \kappa u_\epsilon^m\big)\big) \leqslant 0.$$

Proof. First, note that

$$\int_\Omega \big(F_T^m, s\big(\gamma u_\epsilon^m, \kappa u_\epsilon^m\big)\big)_{\mathbb{R}^2} = 0, \quad \forall s \in \mathbb{R},$$

and this can be proved by noting that $F_T^m = -\Delta\Phi_T^m - A\Phi_T^m$ and $\operatorname{supp}\Phi_T^m \cap \operatorname{supp}(\gamma u_\epsilon^m, \kappa u_\epsilon^m) = \emptyset$.

In the next estimate, we remark the importance of separating the supports of Φ_T^m and $(\gamma u_\epsilon^m, \kappa u_\epsilon^m)$, which makes the following task really easier. That is, because of the equality:

$$\int_\Omega H\big(\big(R\gamma u_\epsilon^m + \varphi_T^m\big)_+, \big(R\kappa u_\epsilon^m + \psi_T^m\big)_+\big) = R^{2^*} H(\gamma,\kappa) \int_\Omega \big(u_\epsilon^m\big)^{2^*},$$

we see that

$$\begin{aligned} J\big(R\big(\gamma u_\epsilon^m, \kappa u_\epsilon^m\big)\big) &\leqslant C\frac{R^2}{2}\int_\Omega \big|\nabla u_\epsilon^m\big|^2 - CR^{2^*}\int_\Omega (u_\epsilon^m)^{2^*} + CR \\ &\leqslant CR + CR^2 - CR^{2^*}. \end{aligned} \tag{5.2}$$

Then, we can choose R_0 sufficiently large to have $J(R(\gamma u_\epsilon^m, \kappa u_\epsilon^m)) \leqslant 0$ for each $R \geqslant R_0$. $\square$

Now we can define

$$\bar{c} = \inf_{\upsilon\in\Upsilon} \sup_{W\in\upsilon(E)} J(W),$$

where $\Upsilon = \{\upsilon \in C(E,E)\colon\ \upsilon(0) = 0 \text{ and } \upsilon(R(\gamma u_\epsilon^m, \kappa u_\epsilon^m)) = R(\gamma u_\epsilon^m, \kappa u_\epsilon^m)\}$, R given by Lemma 5.

The Mountain Pass Theorem assures the existence of a *PS*-sequence for the functional J on the mini–max level $\bar{c}$. In other words, we have a sequence (V_n) such that $J(V_n) \to \bar{c}$ and $J'(V_n) \to 0$.

The goal now is to prove that (V_n) converges to a solution for problem (3.6). We begin by proving that such a sequence has to be bounded.

Proposition 1. *The PS-sequence (V_n) obtained using Lemmas* 4 *and* 5 *is bounded in E.*

Proof. Denote $V_n = (v_{1,n}, v_{2,n})$. A direct calculation with the aide of relations ((4.3) for H and G, (H_3), (G_3)) gives us

$$\begin{aligned} J(V_n) - \frac{1}{2}J'(V_n)V_n = {}& \frac{2}{N-2}\int_\Omega H\big(\big(V_n + \Phi_T^m\big)_+\big) - \frac{1}{2}\int_\Omega H_u\big(\big(V_n + \Phi_T^m\big)_+\big)\varphi_T^m \\ & - \frac{1}{2}\int_\Omega H_v\big(\big(V_n + \Phi_T^m\big)_+\big)\psi_T^m + \left(\frac{p}{2} - 1\right)\int_\Omega G\big(\big(V_n + \Phi_T^m\big)_+\big) \\ & - \frac{1}{2}\int_\Omega G_u\big(\big(V_n + \Phi_T^m\big)_+\big)\varphi_T^m - \frac{1}{2}\int_\Omega G_v\big(\big(V_n + \Phi_T^m\big)_+\big)\psi_T^m \\ & - \frac{1}{2}\int_\Omega \big(F_T - F_T^m, V_n\big)_{\mathbb{R}^2}. \end{aligned}$$

Note, however, that $\Phi_T^m < (0,0)$. Therefore, since $|J(V_n)| \leqslant C$ and $|J'(V_n)V_n| \leqslant C\|V_n\|$, we have, using ((4.2), (H_1), (G_1)),

$$\int_\Omega H\big(\big(V_n + \Phi_T^m\big)_+\big) + \int_\Omega G\big(\big(V_n + \Phi_T^m\big)_+\big) \leqslant C_1 + C_2\|V_n\|. \tag{5.3}$$

In the same way, we also get the following estimates

$$\begin{aligned} &-\int_\Omega H_u\big(\big(V_n + \Phi_T^m\big)_+\big)\varphi_T^m \leqslant C_1 + C_2\|V_n\|; \\ &-\int_\Omega H_v\big(\big(V_n + \Phi_T^m\big)_+\big)\psi_T^m \leqslant C_1 + C_2\|V_n\|; \\ &-\int_\Omega G_u\big(\big(V_n + \Phi_T^m\big)_+\big)\varphi_T^m \leqslant C_1 + C_2\|V_n\|; \\ &-\int_\Omega G_v\big(\big(V_n + \Phi_T^m\big)_+\big)\psi_T^m \leqslant C_1 + C_2\|V_n\|. \end{aligned} \tag{5.4}$$

Now we just need to see that

$$\begin{aligned}\left(1-\frac{\mu_2}{\lambda_1}\right)\|V\|^2 \leqslant{} & J'(V_n)V_n + 2^*\int_\Omega H\big(\big(V_n+\Phi_T^m\big)_+\big) - \int_\Omega H_u\big(\big(V_n+\Phi_T^m\big)_+\big)\varphi_T^m \\ & - \int_\Omega H_v\big(\big(V_n+\Phi_T^m\big)_+\big)\psi_T^m - \int_\Omega G_u\big(\big(V_n+\Phi_T^m\big)_+\big)\varphi_T^m \\ & - \int_\Omega G_v\big(\big(V_n+\Phi_T^m\big)_+\big)\psi_T^m + p\int_\Omega G\big(\big(V_n+\Phi_T^m\big)_+\big) + C_1 + C_2\|V_n\|.\end{aligned}$$

Finally, by (5.3) and (5.4) we get

$$\|V_n\|^2 \leqslant C_1 + C_2\|V_n\|,$$

completing the proof. □

As usual, since V_n is bounded we may now suppose that (eventually passing to a subsequence)

$$V_n \rightharpoonup V \in E \quad \text{and} \quad \|V_n - V\| \text{ is convergent.}$$

Using the fact that H_u and H_v are (2^*-1)-homogeneous and G_u and G_v are $(p-1)$-homogeneous, it is a standard procedure to check that $V=(v_1,v_2)$ must be a solution to problem (3.6). But we still have to assure that $V \neq \Phi_T - \Phi_T^m$.

A first step towards our goal is the following

Lemma 6. *Let $K := \lim_{n\to\infty}\|V_n - V\|^2$. Then*

$$J(V) + \frac{K}{N} = \bar{c}. \tag{5.5}$$

Moreover, if $K>0$ then $K \geqslant (1/2^*)^{(N-2)/2}S_H^{N/2}$, where S_H is given by (4.8).

Proof. First let us state some preliminary facts. All of them are consequences of $V_n \rightharpoonup V$:

(1) We will use a result due to de Morais Filho and Souto [10] which is an extension of the Brezis–Lieb Lemma (see [2]) for homogeneous functions. In our case it assures that

$$\int_\Omega H\big(\big(V_n+\Phi_T^m\big)_+\big) - \int_\Omega H\big(\big(V+\Phi_T^m\big)_+\big) = \int_\Omega H\big((V_n-V)_+\big) + o(1).$$

(2) Since E is a Hilbert Space, we have

$$\|V_n\|^2 = \|V_n - V\|^2 + \|V\|^2 + o(1).$$

(3) Since $p<2^*$ we also get

$$\int_\Omega G\big(\big(V_n+\Phi_T^m\big)_+\big) = \int_\Omega G\big(\big(V+\Phi_T^m\big)_+\big) + o(1).$$

(4) By the Dominated Convergence Theorem and recalling that H_u and H_v are (2^*-1)-homogeneous and G_u and G_v are $(p-1)$-homogeneous we obtain

$$\int_\Omega H_u\big(\big(V_n+\Phi_T^m\big)_+\big)\varphi_T^m = \int_\Omega H_u\big(\big(V+\Phi_T^m\big)_+\big)\varphi_T^m + o(1);$$

$$\int_\Omega H_v\big(\big(V_n+\Phi_T^m\big)_+\big)\psi_T^m = \int_\Omega H_v\big(\big(V+\Phi_T^m\big)_+\big)\psi_T^m + o(1);$$

$$\int_\Omega G_u\big(\big(V_n+\Phi_T^m\big)_+\big)\varphi_T^m = \int_\Omega G_u\big(\big(V+\Phi_T^m\big)_+\big)\varphi_T^m + o(1);$$

$$\int_\Omega G_v\big(\big(V_n+\Phi_T^m\big)_+\big)\psi_T^m = \int_\Omega G_v\big(\big(V+\Phi_T^m\big)_+\big)\psi_T^m + o(1).$$

Now, since $J'(V_n) \to 0$, we get

$$\|V_n\|^2 - \int_\Omega (AV_n, V_n)_{\mathbb{R}^2} + \int_\Omega H_u\big((V_n + \Phi_T^m)_+\big)\varphi_T^m + \int_\Omega H_v\big((V_n + \Phi_T^m)_+\big)\psi_T^m$$
$$- 2^* \int_\Omega H\big((V_n + \Phi_T^m)_+\big) - p \int_\Omega G\big((V_n + \Phi_T^m)_+\big) + \int_\Omega G_u\big((V_n + \Phi_T^m)_+\big)\varphi_T^m$$
$$+ \int_\Omega G_v\big((V_n + \Phi_T^m)_+\big)\psi_T^m - \int_\Omega \big(F_T - F_T^m, V_n\big)_{\mathbb{R}^2} = o(1).$$

Employing (1)–(4) and $J'(V)V = 0$ in the above equation, we see that

$$\|V_n - V\|^2 = 2^* \int_\Omega H\big((V_n - V)_+\big) + o(1). \tag{5.6}$$

Similarly, from $J(V_n) \to \bar{c}$, (1)–(3),

$$\bar{c} + o(1) = J(V) + \frac{1}{2}\|V_n - V\|^2 - \int_\Omega H\big((V_n - V)_+\big). \tag{5.7}$$

And we get (5.5) from (5.6) and (5.7).

To conclude, suppose $K > 0$. Then, by (5.6),

$$\|V_n - V\|^2 \geqslant S_H\left(\int_\Omega H\big((V_n - V)_+\big)\right)^{\frac{N-2}{N}} = S_H\left(\frac{1}{2^*}\|V_n - V\|^2 + o(1)\right)^{\frac{N-2}{N}}$$

and taking $n \to \infty$ we reach $K \geqslant (1/2^*)^{(N-2)/2} S_H^{N/2}$. $\square$

From now on, as in [4], it will be convenient to take $0 < d < 1$ (to be chosen precisely later) and make $\epsilon^d = 1/2m$. This will allow us to simplify all the estimates needed ahead; note that this choice of ϵ is necessary due to Lemma 3 items (4)–(6).

Lemma 7. *Fix $\epsilon^d = 1/2m$. Then*

$$J\big(\Phi_T - \Phi_T^m\big) \leqslant C\epsilon^{dN}.$$

Proof. One only needs to see that

$$J\big(\Phi_T - \Phi_T^m\big) = -\frac{1}{2}\left(\big\|\Phi_T - \Phi_T^m\big\|^2 - \int_\Omega \big(A\big(\Phi_T - \Phi_T^m\big), \Phi_T - \Phi_T^m\big)_{\mathbb{R}^2}\right).$$

The conclusion will follow from (2.4) and (4.1). $\square$

Let $e > 0$ be such that

$$\frac{1}{e} + \frac{1}{r} + \frac{1}{2} = 1, \tag{5.8}$$

where $r > N$ is given in (2.2). This implies that $2 < e < 2^*$.

Now, for the conclusion we will need one last lemma concerning an estimate of the mini–max level. It reads

Lemma 8.

(i) *Suppose $N \geqslant 6$. If $e/N < d < 1 - 2/(N-2)$ then*

$$\bar{c} < \frac{(1/2^*)^{(N-2)/2}}{N} S_H^{N/2} - C\epsilon^2.$$

(ii) *Suppose $N = 3, 4, 5$. If $2^* - p < d < p/2 - 2/(N-2)$ then*

$$\bar{c} < \frac{(1/2^*)^{(N-2)/2}}{N} S_H^{N/2} - C\epsilon^{N-(N-2)p/2}.$$

Proof. Because of (5.2), for each ϵ we can choose $s_\epsilon > 0$ such that

$$\sup_{s\geqslant 0} J\big(s\big(\gamma u_\epsilon^m, \kappa u_\epsilon^m\big)\big) = J\big(s_\epsilon\big(\gamma u_\epsilon^m, \kappa u_\epsilon^m\big)\big).$$

Moreover, one can prove that $s_\epsilon \geqslant l$ for some $l > 0$ and for all $\epsilon > 0$.

Therefore, we only need to prove that

$$J\big(s_\epsilon\big(\gamma u_\epsilon^m, \kappa u_\epsilon^m\big)\big) < \begin{cases} \frac{(1/2^*)^{(N-2)/2}}{N} S_H^{N/2} - C\epsilon^2 & \text{if } N \geqslant 6; \\ \frac{(1/2^*)^{(N-2)/2}}{N} S_H^{N/2} - C\epsilon^{N-(N-2)p/2} & \text{if } N = 3, 4, 5. \end{cases}$$

So, fix $\epsilon > 0$. Using Lemma 3 items (5) and (6), (5.1) and relation (5.8) we have

$$\begin{aligned} J\big(s_\epsilon\big(\gamma u_\epsilon^m, \kappa u_\epsilon^m\big)\big) &\leqslant \frac{s_\epsilon^2}{2}\big(\gamma^2+\kappa^2\big)\int_\Omega \big|\nabla u_\epsilon^m\big|^2 - s_\epsilon^{2^*} H(\gamma,\kappa)\int_\Omega \big(u_\epsilon^m\big)^{2^*} \\ &\quad + s_\epsilon\bigg(\int_{B_{1/m}} |\gamma f_1 + \kappa f_2| u_\epsilon^m\bigg) - C\big(\big\|u_\epsilon^m\big\|_2^2 + G(\gamma,\kappa)\big\|u_\epsilon^m\big\|_p^p\big) \\ &\leqslant \bigg(\frac{s_\epsilon^2}{2} - s_\epsilon^{2^*} H(\gamma,\kappa)\bigg)\big[S^{N/2} + O\big((\epsilon m)^{N-2}\big)\big] \\ &\quad + C|B_{1/m}|^{1/e}\big\|u_\epsilon^m\big\|_2 - C\big(\big\|u_\epsilon^m\big\|_2^2 + G(\gamma,\kappa)\big\|u_\epsilon^m\big\|_p^p\big). \end{aligned}$$

Now let

$$f(r) = \frac{r^2}{2} - r^{2^*} H(\gamma,\kappa).$$

One can prove that f attains its maximum at

$$r_0 = \bigg(\frac{1}{2^* H(\gamma,\kappa)}\bigg)^{(N-2)/4}.$$

Therefore, because of the choice of γ, κ in (5.1),

$$f(s_\epsilon) \leqslant f(r_0) = \frac{1}{N(2^* H(\gamma,\kappa))^{(N-2)/2}} = \frac{1}{N}\bigg(\frac{1}{2^*}\bigg)^{\frac{N-2}{2}}\bigg(\frac{1}{M_{\overline{H}}}\bigg)^{\frac{N}{2}}.$$

Using this last inequality in the above estimate on $J(s_\epsilon(\gamma u_\epsilon^m, \kappa u_\epsilon^m))$, and recalling Lemma 2 we get

$$J\big(s_\epsilon\big(\gamma u_\epsilon^m, \kappa u_\epsilon^m\big)\big) \leqslant \frac{(1/2^*)^{(N-2)/2} S_H^{N/2}}{N} + C_1(\epsilon m)^{N-2} + C_2 m^{-N/e}\big\|u_\epsilon^m\big\|_2 - C\big(\big\|u_\epsilon^m\big\|_2^2 + G(\gamma,\kappa)\big\|u_\epsilon^m\big\|_p^p\big). \tag{5.9}$$

Now we must treat $N \geqslant 6$ and $N = 3, 4, 5$ separately.

If $N \geqslant 6$.

We will not need $G(\gamma,\kappa)\|u_\epsilon^m\|_p^p$ in (5.9). Recall Lemma 3 items (3) and (4) (we choose $k = 2$) to see that $K_1\epsilon^2 \leqslant \|u_\epsilon^m\|_2^2 \leqslant K_2\epsilon^2$, where we recall that we are working with $\epsilon^d = 1/2m$. Putting these relations in (5.9) will give us

$$J\big(s_\epsilon\big(\gamma u_\epsilon^m, \kappa u_\epsilon^m\big)\big) \leqslant \frac{(1/2^*)^{(N-2)/2} S_H^{N/2}}{N} + C_1\epsilon^{(1-d)(N-2)} + C_2\epsilon^{1+dN/e} - C\epsilon^2.$$

Therefore, taking $e/N < d < 1 - 2/(N-2)$ (this is possible only if $N \geqslant 6$) we have $(1-d)(N-2), 1 + dN/e > 2$ which means that for ϵ sufficiently small we can choose $C > 0$ such that $J(s_\epsilon(\gamma u_\epsilon^m, \kappa u_\epsilon^m)) \leqslant (1/2^*)^{(N-2)/2} S_H^{N/2}/N - C\epsilon^2$, proving (i).

If $N = 3, 4, 5$.

In this case, we drop the term $\|u_\epsilon^m\|_2^2$ in (5.9). Note that the additional hypothesis (G_4) assures $G(\gamma,\kappa) > 0$.

For dimensions 3, 4 we cannot use Lemma 3 item (3). Nevertheless, we will just need that $\|u_\epsilon^m\|_2 \leqslant C$. Using this time Lemma 3 item (4) with $k = p$, one gets

$$J\big(s_\epsilon\big(\gamma u_\epsilon^m, \kappa u_\epsilon^m\big)\big) \leqslant \frac{(1/2^*)^{(N-2)/2} S_H^{N/2}}{N} + C_1\epsilon^{(1-d)(N-2)} + C_2\epsilon^{dN/e} - C\epsilon^{N-(N-2)p/2}.$$

This proves (ii) only if $(1-d)(N-2), dN/e > N - (N-2)p/2$, that is, we have to pick d such that

$$2^* - p < d < \frac{p}{2} - \frac{2}{N-2}$$

and this can be done due to condition (G_4). □

Proof of Theorem 1. All that remains to prove is $V \neq \Phi_T - \Phi_T^m$, where V is the weak limit of the *PS*-sequence in the minimax level $\bar{c}$. We observe that if we had $K = 0$ (K given in Lemma 6), it would mean that (V_n) would converge strongly to V in E and so $J(V) = \bar{c}$. Therefore, we could choose small ϵ to have $J(V) = \bar{c} \geqslant \delta > C\epsilon^{dN} \geqslant J(\Phi_T - \Phi_T^m)$, by Lemma 7, which would prove that $V \neq \Phi_T - \Phi_T^m$ and we would be done. Therefore, suppose $K > 0$. If $V = \Phi_T - \Phi_T^m$ we have, by Lemmas 6 and 7

$$\frac{(1/2^*)^{(N-2)/2} S_H^{N/2}}{N} - C\epsilon^{dN} \leqslant \frac{K}{N} + J(V) = \bar{c}$$

but Lemma 8 says that

$$\bar{c} < \begin{cases} \frac{(1/2^*)^{(N-2)/2} S_H^{N/2}}{N} - C\epsilon^2 & \text{if } N \geqslant 6; \\ \frac{(1/2^*)^{(N-2)/2} S_H^{N/2}}{N} - C\epsilon^{N-(N-2)p/2} & \text{if } N = 3, 4, 5. \end{cases}$$

And this is impossible because of the choice of d in Lemma 8. □

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